Problem 1.

Demonstrate (check for the properties) that the following function is an inner product in $\mathbf{R}^3$. (Call $\mathbf{R}^3$ with this inner product Moorean 3-space for the all of these problems following) Let $u=(u_1,u_2,u_3)$ and $v=(v_1,v_2,v_3)$. Then:

$\langle u,v\rangle = uAv^T$**, where A is the matrix** $A=\begin{pmatrix}2&0&0\\0&1&0\\0&0&2\end{pmatrix}$.

Solution:
First we need to review the definition of the inner product:

An inner product is a generalization of the dot product. In a vector space, it is a way to multiply vectors together, with the result of this multiplication being a scalar. More precisely, for a real vector space, an inner product $\langle\cdot,\cdot\rangle$ satisfies the following four properties. Let *u*, *v*, and *w* be vectors and α be a scalar, then:

1. $\langle u+v,w\rangle=\langle u,w\rangle+\langle v,w\rangle$.
2. $\langle \alpha v,w\rangle=\alpha\langle v,w\rangle$.
3. $\langle v,w\rangle=\langle w,v\rangle$.
4. $\langle v,v\rangle\geq 0$ and equal if and only if $v=0$.

Let's simply check these properties for our given function. If we prove that it satisfies the four properties listed above, we'll thus show that it is an inner product indeed.

1. $\langle u+v,w\rangle=(u+v)Aw^T=uAw^T+vAw^T=\langle u,w\rangle+\langle v,w\rangle$
2. $\langle \alpha v,w\rangle=(\alpha v)Aw^T=\alpha vAw^T=\alpha\left(vAw^T\right)=\alpha\langle v,w\rangle$
3. $\langle v,w\rangle=vAw^T=$

$$=(v_1,v_2,v_3)\cdot\begin{pmatrix}2&0&0\\0&1&0\\0&0&2\end{pmatrix}\cdot\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}=(2v_1,v_2,2v_3)\cdot\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}=2v_1w_1+v_2w_2+2v_3w_3=$$

$$=(2w_1,w_2,2w_3)\cdot\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=(w_1,w_2,w_3)\cdot\begin{pmatrix}2&0&0\\0&1&0\\0&0&2\end{pmatrix}\cdot\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=$$

$$=wAv^T=\langle w,v\rangle.$$

4. $\langle v,v\rangle=vAv^T=$

$$= (v_1, v_2, v_3) \cdot \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = (2v_1, v_2, 2v_3) \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 2v_1^2 + v_2^2 + 2v_3^2 > 0,$$

And it equals 0 only if v = (0,0,0).
So, we have checked all the properties and showed that they stand true. So, the function $\langle u, v \rangle = uAv^T$ is an inner product indeed.

Problem 2

Find $\|x\|$ in Moorean 3-space $\mathbf{R}^3$ (see problem 1) where $x = (1, -3, -2)$.

Solution.
Every inner product space is a normed vector space with the norm being defined by

$$\|v\| = \sqrt{\langle v, v \rangle}.$$

In Moorean 3-space $\mathbf{R}^3$ we get:

$$\langle v, v \rangle = vAv^T =$$

$$= (v_1, v_2, v_3) \cdot \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = (2v_1, v_2, 2v_3) \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 2v_1^2 + v_2^2 + 2v_3^2.$$

Substituting $x = (1, -3, -2)$ we get

$$\langle x, x \rangle = 2x_1^2 + x_2^2 + 2x_3^2 = 2 \cdot 1 + 9 + 2 \cdot 4 = 19.$$

So, $\|x\| = \sqrt{19}$.

Problem 3

Prove that if u and v are given non zero vectors in the arbitrary inner-product space V, and are such that $\langle u, v \rangle = 0$ then $\{u, v\}$ is linearly independent subset of V.

Two vectors u and v are linearly independent if the linear combination αu + βv equals 0 only in case α = β = 0. So we take their linear combination and we try to prove that the coefficients α and β are equal 0. Also, we will apply the properties 1-4 of the inner product, which were listed in the first problem.
It is obvious that 0 = <0, u>
Then it implies 0 = < αu + βv, u> = < αu,u> + <βv, u> = α< u,u> + β<v, u> = α< u,u> (because <u,v> = <v, u> =0). So α< u,u>=0 and we know that due to the property #4: < u,u> > 0 in case u is a non-zero vector. So the only possibility is that α =0.
The same thing with β:
0 = < αu + βv, v> = < αu,v> + <βv, v> = α< u,v> + β<v, v> = β<v, v> . This implies β=0.